Lecture 18: November 4

Let us first recall the construction from last time. We were looking at a polarized variation of Hodge structure of weight n on the punctured disk Δ^*. For each $\alpha \in \mathbb{R}$, we have a canonical extension $\tilde{\mathscr{V}}^\alpha$ of the vector bundle $\mathscr{V}$ to a vector bundle on Δ, and we defined

$$\tilde{\mathscr{V}} = \bigcup_{\alpha \in \mathbb{R}} \tilde{\mathscr{V}}^\alpha \subseteq j_* \mathscr{V},$$

and showed that $\tilde{\mathscr{V}}$ is a coherent $\mathscr{D}_\Delta$-module. The subsheaves $\tilde{\mathscr{V}}^\alpha$ define a decreasing filtration by locally free $\mathscr{O}_\Delta$-modules, in the sense that $\alpha \leq \beta$ implies $\tilde{\mathscr{V}}^\beta \subseteq \tilde{\mathscr{V}}^\alpha$. Moreover, we have

$$t \cdot \tilde{\mathscr{V}}^\alpha \subseteq \tilde{\mathscr{V}}^{\alpha+1} \quad \text{and} \quad \partial_t \cdot \tilde{\mathscr{V}}^\alpha \subseteq \tilde{\mathscr{V}}^{\alpha-1}$$

for every $\alpha \in \mathbb{R}$. The filtration $\tilde{\mathscr{V}}^\bullet$ records the information about the monodromy transformation $T \in \mathrm{GL}(V)$, in the following manner. Recall that the fiber of each canonical extension $\tilde{\mathscr{V}}^\alpha$ is canonically identified with the vector space V of flat sections of $\exp^* \mathscr{V}$ on $\mathbb{H}$. Therefore

$$\tilde{\mathscr{V}}^\alpha / \tilde{\mathscr{V}}^{\alpha+1} = \tilde{\mathscr{V}}^\alpha / t \tilde{\mathscr{V}}^\alpha \cong \tilde{\mathscr{V}}^\alpha \big|_0 \cong V.$$

Under this isomorphism, the operator $t\partial_t$ goes to the residue of the logarithmic connection, hence to the endomorphism $R \in \mathrm{End}(V)$. Recall that $R = R_S + R_N$, where R_S has eigenvalues in the interval $[\alpha, \alpha+1)$. Therefore

$$\tilde{\mathscr{V}}^\alpha / \tilde{\mathscr{V}}^{>\alpha} \cong E_\alpha(R_S),$$

and under this isomorphism, the operator $t\partial_t - \alpha$ goes to $R - \alpha = R_N$. In particular, $\tilde{\mathscr{V}}^\alpha / \tilde{\mathscr{V}}^{>\alpha}$ is a finite-dimensional vector space on which the operator $t\partial_t - \alpha$ acts nilpotently.

Local systems. On Δ^*, the local system $\mathscr{V}^\nabla$ of ∇-flat sections of $\mathscr{V}$ is resolved by the de Rham-type complex

$$\mathscr{V} \xrightarrow{\nabla} \Omega^1_{\Delta^*} \otimes_{\mathscr{O}_{\Delta^*}} \mathscr{V}$$

To understand the meromorphic extension $\tilde{\mathscr{V}}$ better, we now investigate the analogous complex

$$\tilde{\mathscr{V}} \xrightarrow{\nabla} \Omega^1_\Delta \otimes_{\mathscr{O}_\Delta} \tilde{\mathscr{V}} \ , \tag{18.1}$$

which is a complex of sheaves of $\mathbb{C}$-vector spaces on Δ. Outside the origin, the complex is of course a resolution of $\mathscr{V}^\nabla$.

Lemma 18.2. *The cohomology sheaves of this complex are $j_*(\mathscr{V}^\nabla)$ in degree 0, and $R^1 j_*(\mathscr{V}^\nabla)$ in degree 1.*

Proof. Since we already know what happens on Δ^*, it suffices to compute the stalks of the two cohomology sheaves at the origin. We have

$$R^k j_*(\mathscr{V}^\nabla)_0 = \lim_{U \ni 0} H^k(U \cap \Delta^*, \mathscr{V}^\nabla).$$

Using a covering of Δ^* by two simply connected open sets, this is computed by the complex

$$V \xrightarrow{T - \mathrm{id}} V$$

and is therefore isomorphic to $\ker(T - \mathrm{id})$ for $k = 0$, and to $\mathrm{coker}(T - \mathrm{id})$ for $k = 1$.

Now let us study the complex in (18.1). Observe that $\Omega^1_\Delta = \mathscr{O}_\Delta dt$, and that $t\colon \tilde{\mathscr{V}} \to \tilde{\mathscr{V}}$ is an isomorphism by construction; the cohomology sheaves of our complex are therefore going to be isomorphic to those of

$$\tilde{\mathscr{V}} \xrightarrow{t\partial_t} \tilde{\mathscr{V}}.$$

The point is that for each $\alpha \in \mathbb{R}$, we now have a subcomplex

$$\tilde{\mathscr{V}}^\alpha \xrightarrow{t\partial_t} \tilde{\mathscr{V}}^\alpha.$$

Since $t\partial_t - \alpha = R_N$ acts nilpotently on $\tilde{\mathscr{V}}^\alpha/\tilde{\mathscr{V}}^{>\alpha} \cong E_\alpha(R_S)$, the complex

$$\tilde{\mathscr{V}}^\alpha/\tilde{\mathscr{V}}^{>\alpha} \xrightarrow{t\partial_t} \tilde{\mathscr{V}}^\alpha/\tilde{\mathscr{V}}^{>\alpha}$$

is exact except for $\alpha = 0$. This implies pretty easily that the inclusion

$$\begin{tikzcd} \tilde{\mathscr{V}}^0 \arrow[r, "t\partial_t"] \arrow[d, hook] & \tilde{\mathscr{V}}^0 \arrow[d, hook] \\ \tilde{\mathscr{V}} \arrow[r, "t\partial_t"] & \tilde{\mathscr{V}} \end{tikzcd}$$

induces isomorphisms on cohomology. Moreover, the surjection

$$\begin{tikzcd} \tilde{\mathscr{V}}^0 \arrow[r, "t\partial_t"] \arrow[d] & \tilde{\mathscr{V}}^0 \arrow[d] \\ \tilde{\mathscr{V}}^0/\tilde{\mathscr{V}}^{>0} \arrow[r, "t\partial_t"] & \tilde{\mathscr{V}}^0/\tilde{\mathscr{V}}^{>0} \end{tikzcd}$$

induces isomorphisms of the stalks of the cohomology sheaves the origin; the reason is that $\tilde{\mathscr{V}}^{\alpha+1} = t\tilde{\mathscr{V}}^\alpha$, and so if a section on a neighborhood of the origin belongs to every $\tilde{\mathscr{V}}^\alpha$, then it must be zero by Krull's lemma. This reduces the problem to computing the cohomology of the complex

$$E_0(R_S) \xrightarrow{R_N} E_0(R_S).$$

Recall that $T = e^{2\pi i R}$ is the monodromy transformation. In degree 0, we get

$$\{\, v \in V \mid R_S v = R_N v = 0 \,\} = \{\, v \in V \mid Tv = v \,\},$$

which is exactly the stalk of the sheaf $j_* \mathscr{V}^\nabla$. In degree 1, we get

$$E_0(R_S)/R_N E_0(R_S) \cong V/TV,$$

and you can check that this is isomorphic to $\operatorname{coker}(T - \mathrm{id})$, hence to the stalk of the sheaf $R^1 j_* \mathscr{V}^\nabla$. $\square$

Recall from last time that $\tilde{\mathscr{V}} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{-1}$. The proof breaks down for $\tilde{\mathscr{V}}^{>-1}$, and so we can get a smaller $\mathscr{D}$-module by considering the submodule

$$\mathcal{M} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{>-1} \subseteq \tilde{\mathscr{V}}.$$

It is called the *minimal extension* of the vector bundle with connection, for reasons that will become clear in a moment. We have an induced filtration

$$V^\alpha \mathcal{M} = \tilde{\mathscr{V}}^\alpha \cap \mathcal{M},$$

and by construction, $V^\alpha \mathcal{M} = \tilde{\mathscr{V}}^\alpha$ for $\alpha > -1$.

Lemma 18.3. *We have $V^{-1}\mathcal{M} = \partial_t \cdot V^0 \mathcal{M} + V^{>-1}\mathcal{M}$.*

Proof. One inclusion is obvious. For the other one, suppose that we have a local section $s \in V^{-1}\mathcal{M}$. Then $s \in \tilde{\mathscr{V}}^{-1}$ and also $s \in \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{>-1}$, and so

$$s = \partial_t s' + s''$$

for certain local sections $s' \in \mathcal{M}$ and $s'' \in \tilde{\mathscr{V}}^{>-1}$. I claim that this forces $s' \in \tilde{\mathscr{V}}^0$. The reason is that

$$\partial_t \colon \tilde{\mathscr{V}}^\alpha / \tilde{\mathscr{V}}^{>\alpha} \to \tilde{\mathscr{V}}^{\alpha-1} / \tilde{\mathscr{V}}^{>(\alpha-1)}$$

is an isomorphism for every $\alpha \neq 0$. Now if $s' \in \tilde{\mathscr{V}}^\alpha$ for some $\alpha < 0$, then we can project the identity $\partial_t s' = s - s'' \in \tilde{\mathscr{V}}^{-1}$ into $\tilde{\mathscr{V}}^{\alpha-1} / \tilde{\mathscr{V}}^{>(\alpha-1)}$, and conclude that $s' \in \tilde{\mathscr{V}}^{>\alpha}$, hence $s' \in \tilde{\mathscr{V}}^{\alpha+\varepsilon}$ for some $\varepsilon > 0$. Repeating this argument finally many times eventually yields $s' \in \tilde{\mathscr{V}}^0$. □

In analogy with what we did for $\tilde{\mathscr{V}}$, let us define

$$\operatorname{gr}_V^\alpha \mathcal{M} = V^\alpha \mathcal{M} / V^{>\alpha} \mathcal{M},$$

which is a subspace of $\tilde{\mathscr{V}}^\alpha / \tilde{\mathscr{V}}^{>\alpha}$, hence again a finite-dimensional vector space. For $\alpha > -1$, the inclusion is an isomorphism, hence

$$\operatorname{gr}_V^\alpha \mathcal{M} \cong E_\alpha(R_S),$$

with $t\partial_t - \alpha$ acting as the nilpotent operator R_N. For $\alpha = -1$, the lemma shows that

$$\partial_t \colon \operatorname{gr}_V^0 \mathcal{M} \to \operatorname{gr}_V^{-1} \mathcal{M}$$

is surjective. Note that $t \colon \mathcal{M} \to \mathcal{M}$ is injective (because this is true on the larger $\mathscr{D}_\Delta$-module $\tilde{\mathscr{V}}$.

Exercise 18.1. Check that the de Rham-type complex

$$\mathcal{M} \to \Omega^1_\Delta \otimes_{\mathscr{O}_\Delta} \mathcal{M}$$

only has cohomology in degree 0, and that the 0-th cohomology sheaf is isomorphic to $j_* \mathscr{V}^\nabla$. By going to the smaller $\mathscr{D}$-module $\mathcal{M}$, we have therefore eliminated the cohomology sheaf in degree 1.

When we discuss the polarization, we will see that there are other good reasons for working with $\mathcal{M}$ instead of with the meromorphic extension $\tilde{\mathscr{V}}$.

The Hodge filtration. The next step is to extend the Hodge bundles $F^p\mathscr{V}$ to a filtration of $\mathcal{M}$. In $\mathscr{D}$-module theory, it is customary to study $\mathscr{D}$-modules (which are typically not coherent as $\mathscr{O}$-modules) with the help of increasing filtrations by coherent $\mathscr{O}$-modules. We should therefore convert the decreasing Hodge filtration into an increasing filtration by setting

$$F_p \mathscr{V} \underset{\text{def}}{=} F^{-p} \mathscr{V} \subseteq \mathscr{V}.$$

The Griffiths transversality condition reads

$$\partial_t \cdot F_p \mathscr{V} = \nabla_{\partial_t} F_p \mathscr{V} \subseteq F_{p+1} \mathscr{V},$$

which means that the filtration $F_\bullet \mathscr{V}$ is compatible with the action by differential operators. How can we get a suitable filtration of $\mathcal{M}$? Since $\mathcal{M} \subseteq j_* \mathscr{V}$, one could try to use

$$F_p \mathcal{M} = \mathcal{M} \cap j_* F_p \mathscr{V} \subseteq j_* \mathscr{V},$$

but the trouble is that these sheaves will generally not be coherent over $\mathscr{O}_\Delta$. So we have to proceed more carefully. Recall from Theorem 9.1 that the Hodge bundles extend to holomorphic subbundles of any canonical extension; let us denote these bundles by the symbol

$$F_p \tilde{\mathscr{V}}^\alpha \quad \text{and} \quad F_p \tilde{\mathscr{V}}^{>\alpha}.$$

Note. The fact that $F_p \tilde{\mathscr{V}}^\alpha$ is a subbundle means that the quotient sheaf $\tilde{\mathscr{V}}^\alpha / F_p \tilde{\mathscr{V}}^\alpha$ is locally free; you should convince yourself that, therefore,

$$F_p \tilde{\mathscr{V}}^\alpha = \tilde{\mathscr{V}}^\alpha \cap j_*(F_p \mathscr{V}) \subseteq j_* \mathscr{V}.$$

From this point of view, Theorem 9.1 is asserting that the $\mathscr{O}$-module on the right-hand side is coherent.

Since $\mathcal{M} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{>-1}$, and since we would like the filtration on $\mathcal{M}$ to be compatible with the action by differential operators, we now define

$$F_p \mathcal{M} = \sum_{j=0}^{\infty} \partial_t^j \cdot F_{p-j} \tilde{\mathscr{V}}^{>-1}.$$

We have $F_p \mathscr{V} = 0$ for $p \leq p_0$, and therefore also $F_p \mathcal{M} = 0$ for $p \leq p_0$. For the same reason, the sum on the right-hand side is actually finite, and so each $F_p \mathcal{M}$ is finitely generated as an $\mathscr{O}_\Delta$-module, and therefore coherent. The filtration is also "good" in the sense of $\mathscr{D}$-module theory, which means the following.

Lemma 18.4. *We have $\partial_t \cdot F_p \mathcal{M} \subseteq F_{p+1} \mathcal{M}$, with equality for $p \gg 0$.*

Proof. We only need to prove the second half of the assertion; the first is obvious from the definition. For any $p \in \mathbb{Z}$, we have

$$F_{p+1} \mathcal{M} = F_{p+1} \tilde{\mathscr{V}}^{>-1} + \sum_{j=0}^{\infty} \partial_t^{j+1} \cdot F_{(p+1)-(j+1)} \tilde{\mathscr{V}}^{>-1} = F_{p+1} \tilde{\mathscr{V}}^{>-1} + \partial_t \cdot F_p \mathcal{M}.$$

For $p \gg 0$, we have $F_p \tilde{\mathscr{V}}^{>-1} = F_{p+1} \tilde{\mathscr{V}}^{>-1} = \tilde{\mathscr{V}}^{>-1}$. Since we already know that $\partial_t \colon \tilde{\mathscr{V}}^{>-1} \to \tilde{\mathscr{V}}^{>-1}$ is surjective, this gives us

$$F_{p+1} \tilde{\mathscr{V}}^{>-1} = \partial_t \cdot F_p \tilde{\mathscr{V}}^{>-1} \subseteq \partial_t \cdot F_p \mathcal{M},$$

and therefore $F_{p+1} \mathcal{M} = \partial_t \cdot F_p \mathcal{M}$. □

In conclusion, we obtain a coherent $\mathscr{D}_\Delta$-module $\mathcal{M}$, together with an increasing filtration by coherent $\mathscr{O}_\Delta$-modules $F_p \mathcal{M}$. The filtration is compatible with differential operators, and if we restrict $(\mathcal{M}, F_\bullet \mathcal{M})$ to the punctured disk, we get back $(\mathscr{V}, F_\bullet \mathscr{V})$.

The polarization. The last thing to do is to extend the polarization

$$h_{\mathscr{V}} \colon \mathscr{V} \otimes_{\mathbb{C}} \overline{\mathscr{V}} \to \mathscr{C}^\infty_{\Delta^*}$$

to some kind of pairing on $\mathcal{M}$. Here again, we need to go from C^∞-functions to a larger class of functions, to account for the singularity at the origin. A clue to what sort of functions to allow comes from our computation of the pairing in Lecture 9. Back then, we found that in the trivialization $\mathscr{O}_\Delta \otimes_{\mathbb{C}} V \cong \tilde{\mathscr{V}}^{>-1}$, the polarization takes the form

$$h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') = \sum_{-1<\alpha\leq 0} |t|^{2\alpha} \sum_{j=0}^{\infty} \frac{L(t)^j}{j!} (-1)^j h\big(v'_\alpha, R_N^j v''_\alpha\big).$$

Here $v', v'' \in V$ are two vectors, and v'_α, v''_α are the components with respect to the eigenspace decomposition

$$V = \bigoplus_{-1<\alpha\leq 0} E_\alpha(R_S),$$

where $R = R_S + R_N$ is the Jordan decomposition of the residue $R = \operatorname{Res}_0 \nabla$. Notice that the functions $|t|^{2\alpha} L(t)^j$ in the above formula are all locally integrable near the origin; since $|t|^{-2}$ is *not* locally integrable, this property would fail if we used $\tilde{\mathscr{V}}^{-1}$.

Since $\mathcal{M} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{>-1}$, we also need to allow derivatives, and so it is natural to work with distributions: every locally integrable function defines a distribution, and distributions can be differentiated to any order.

Definition 18.5. A distribution on a 1-dimensional complex manifold X is a continuous linear functional on the space $A_c^{1,1}(X,\mathbb{C})$ of compactly supported smooth $(1,1)$-forms.

We denote by $\mathrm{Db}(X)$ the space of distributions on X. Given a distribution $D \in \mathrm{Db}(X)$ and a compactly supported $(1,1)$-form φ, we denote by

$$\langle D, \eta \rangle \in \mathbb{C}$$

the complex number obtained by evaluating D on the "test form" η. If t is a local coordinate, we can write η in the form $\varphi dt \wedge d\bar{t}$ for $\varphi \in C_c^\infty(X)$ a compactly supported smooth function.

Example 18.6. Any locally integrable function $f\colon X \to \mathbb{C}$ defines a distribution by

$$\langle f, \eta \rangle = \int_X f\eta.$$

By analogy with this example, people sometimes write

$$\int_X D\,\eta \underset{\mathrm{def}}{=} \langle D, \eta \rangle$$

for the evaluation of D on η.

Derivatives of distributions are defined by formally integrating by parts: in local coordinates, we set

$$\left\langle \partial_t D, \varphi dt \wedge d\bar{t} \right\rangle \underset{\mathrm{def}}{=} -\left\langle D, \frac{\partial \varphi}{\partial t} dt \wedge d\bar{t} \right\rangle$$
$$\left\langle \overline{\partial}_t D, \varphi dt \wedge d\bar{t} \right\rangle \underset{\mathrm{def}}{=} -\left\langle D, \frac{\partial \varphi}{\partial \bar{t}} dt \wedge d\bar{t} \right\rangle.$$

This is consistent with the formula for integration by parts in case D is the distribution defined by a continuously differentiable function. By this formula, $\mathrm{Db}(X)$ becomes a left module over the ring of differential operators on X and its conjugate.

We denote by Db_X the sheaf with $\Gamma(U, \mathrm{Db}_X) = \mathrm{Db}(U)$ for open subsets $U \subseteq X$. This is a left module over the sheaf of differential operators $\mathscr{D}_X$ and its conjugate $\mathscr{D}_{\bar{X}}$ (and the two structures commute).

Back to the problem of extending the polarization to $\mathcal{M}$. Since $|t|^{2\alpha} L(t)^j$ defines a distribution for $\alpha > -1$ and $j \geq 0$, we already have a pairing

$$h_{\mathscr{V}}\colon \tilde{\mathscr{V}}^{>-1} \otimes_{\mathbb{C}} \overline{\tilde{\mathscr{V}}^{>-1}} \to \mathrm{Db}_\Delta\,.$$

Since $\mathcal{M} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{>-1}$, we obtain the desired sesquilinear pairing

$$h_{\mathcal{M}}\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathrm{Db}_\Delta$$

by extending sesquilinearly.